3. KONDAUROV V.I., On equations of an elastic viscoplastic medium with finite strains, Prikl. Mekh. Tekh. Fiz., NO.4, 133-139, 1982.
4. TRUESDELL C., Introductory Course in Rational Continuum Mechanics, Mir, Moscow, 1975.
5. GODUNOV S.K., Elements of Continuum Mechanics, Nauka, Moscow, 1978.
6. TOUPIN R.A., Theories of elasticity with couple-stress. Arch. Ration. Mech. Anal., Vol.17, No.2, 85-112, 1964.
7. LUR'E A.I., Non-linear Elasticity Theory, Nauka, Moscow, 1980.
8. COURANT R., Partial Differential Equations, Mir, Moscow, 1964.
9. THOMAS T., Plastic Flow and Fracture in Solids, Mir, Moscow, 1964.
10. GODUNOV S.K., Equations of Mathematical Physics, Nauka, Moscow, 1971.

Translated by M.D.F.

PMM U.S.S.R., Vol.48, No.3, pp.299-302, 1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

ON THE STABILITY OF ONE-DIMENSIONAL STATIONARY SOLUTIONS OF HYPERBOLIC SYSTEMS OF DIFFERENTIAL EQUATIONS CONTAINING POINTS AT WHICH ONE CHARACTERISTIC VELOCITY BECOMES ZERO*

A.G. KULIKOVSKII and F.A. SLOBODKINA

The stability of the stationary solutions of hyperbolic systems of partial differential equations containing a point at which one of the characteristic velocities becomes zero, is investigated. The functions sought are assumed to be time and coordinate dependent, and their number is arbitrary.

The study of stability carried out below is based on the results obtained in /1, 2/, according to which the behaviour of the unsteady perturbations near the critical point is described by a single non-linear partial differential equation irrespective of the number of equations in the initial system. The equation is written in terms of a function analogous to the Riemann invariant connected with the vanishing characteristic velocity.

The equation is used below to examine all possible cases of continuous solutions of an arbitrary hyperbolic system of equations with continuous and discontinuous right-hand sides, and conditions are formulated under which the growth of perturbations near the critical point at which one of the characteristic velocities becomes zero, leads to the instability of the whole solution in toto. The investigation is carried out taking into account the onset and development of the perturbations connected with other characteristic velocities which have a constant sign within the region considered.

1. Let us consider a hyperbolic system containing an arbitrary number of equations the unknown functions of which depend on the spatial coordinate x and the time t

$$l_j^i(u_k, x)\left[\frac{\partial u_j}{\partial t} + c^i(u_k, x)\frac{\partial u_j}{\partial x}\right] = f^i(u_k, x) \tag{1.1}$$

System (1.1) is written in the characteristic form, $c^i(u_k, x)$ are the characteristic velocities of the system, and repeated lower Latin indices denote summation from 1 to n.

The elements of the matrix l_j^i and the function c^i are assumed to be continuous and differentiable functions of their arguments, and the right-hand sides of (1.1) are assumed to be piecewise continuous and may have first-order discontinuities in some planes $x = \text{const}$. We shall assume that the first-order partial derivatives of $f^i(u_k, x)$ with respect to all arguments exist and are continuous wherever $f^i(u_k, x)$ are defined, except at the points belonging to the surfaces of discontinuity of these functions.

*Prikl.Matem.Mekhan.,48,3,414-419,1984

We shall study the solution of (1.1) on a finite segment of the x axis. The boundary conditions are specified at the ends $x = -L_1$ and $x = L_2$ of this segment.

We shall assume that one of the single characteristic velocities of system (1.1), e.g. $c^1(u_k, x)$, vanishes in the domain of variation of the variables u_k and x, and the remaining velocities ($c^2, c^3, \ldots, c^n$) do not change their signs.

Let us choose a steady solution $u_j = U_j(x)$ of system (1.1) which has a common point with the surface $c^1(u_k, x)=0$ and is continuous in its small neighbourhood. We shall call this point critical and use it as the reference point for the coordinate x and the quantities u_j. By virtue of this choice we have $x = 0$, $U_j = 0$ and $c^1(0, 0, \ldots, 0) = 0$ at the critical points.

The necessary condition for a continuous single-valued solution $U_j(x)$ to exist is, that the function f^1 on the right-hand side of (1.1) must change its sign at the critical point either continuously, or discontinuously /1, 2/. In the first case the critical points represent the singularities of the steady equations of (1.1), and in the second case the derivatives dU_j/dx become infinite at $x = 0$.

As was shown in /1, 2/ for the continuous and discontinuous function f^1, the behaviour of the steady, as well as the non-steady solution, in a fairly small neighbourhood $-\delta \leqslant x \leqslant \delta$ of the cirtical point, can be described approximately by a single first-order differential equation

$$\frac{\partial c}{\partial t} + [c + \varphi(t)]\frac{\partial c}{\partial x} = \gamma + \alpha c + \beta x + F(t) \tag{1.2}$$

Here α, β, γ are constants determined by the form of the initial system (1.1) and $c(x, t)$ is the unknown function assumed small in the region considered. The equation shows that the perturbations of $c(x, t)$ propagate with characteristic velocity $c + \varphi$.

The quantities φ and F can be expressed linearly in terms of the "transient" perturbations v_μ, and are connected with other characteristic quantities of (1.1) whose velocities do not vanish within the region considered.

To solve (1.2) we must know the functions $F(t)$ and $\varphi(t)$, and the boundary conditions defining c at the ends $[-\delta, \delta]$ of the segment, in the cases when the characteristics of (1.2) lie within the segment.

The following expression was obtained in /2/ for v_μ:

$$v_\mu = d_\mu \int_0^x c^*(\xi, t)\, d\xi + p_\mu(t), \quad \mu = 2, 3, \ldots, n \tag{1.3}$$

Here d_μ is a constant depending on the initial system (1.1), p_μ is an arbitrary function of time, and $c^*(x, t) = c(x, t) - C(x)$ are the perturbations of the steady solution $C(x)$ of (1.2). For small δ the first term on the right-hand side of (1.3) is small, therefore v_μ can be regarded as functions of time only. The expression also determines the quantities φ and F which are given in terms of p_μ by the linear expressions with constant coefficients a_μ and b_μ

$$\varphi = \varphi(t) = \sum_\mu a_\mu p_\mu(t), \quad F = F(t) = \sum_\mu b_\mu p_\mu(t) \tag{1.4}$$

From (1.3) it follows that the presence of a perturbation $c^*(x, t)$ on the segment $[-\delta, \delta]$ can give rise to the perturbations v_μ, which it turn affect the solution of (1.2) through the functions $\varphi(t)$ and $F(t)$. Moreover, the perturbations v_μ, propagating over the whole region where the solution of (1.1) is studied, will be reflected from the boundary of this region and from the inhomogeneities of the solution whose stability is being studied. The reflections generate, in general, the perturbations c^*. On arriving at the boundaries of the segment $[-\delta, \delta]$ these perturbations fix the boundary conditions for (1.2).

The present paper studies the interaction between the perturbation c^* and transient perturbations v_μ, and its effect on the behaviour of the unsteady solution in the neighbourhood of the critical point, and on the stability of the steady solutions.

Equations (1.3) show that the transient perturbations v_μ change over the segment $[-\delta, \delta]$ by a quantity proportional to

$$S(t) = \int_{-\delta}^{\delta} c^*(x, t)\, dx$$

As was shown in /1, 2/ using (1.2), the quantity $S(t)$ is given by the equation

$$\frac{dS}{dt} = \alpha S + q, \quad q = q_1 + q_2 \tag{1.5}$$

$$q_1 = -\int_0^{c^*(\delta)} [C(\delta) + \xi]\, d\xi = -\left[Cc^* + \frac{1}{2}c^{*2}\right]_{x=\delta}$$

$$q_2 = \left[Cc^* + \frac{1}{2}c^{*2}\right]_{x=-\delta}$$

The quantities q_1 and q_2 represent the inflow (gain) (or outflow(loss) when they are negative) in the area across the boundaries of the segment $[-\delta, \delta]$. They are determined for the given $C(\pm\delta)$ by the values of $c^*(\delta)$ and $c^*(-\delta)$.

2. As we said before, to study the behaviour of the perturbations we must know the quantities φ, F and q_i. The quantity q_i has to be specified only when the characteristics of (1.2) arrive from the outside at the corresponding boundary of the segment $[-\delta, \delta]$. Let us obtain estimates for φ, F and q_i under some natural general assumptions.

Let the transient perturbations v_μ be generated only when the perturbations $c^*(x, t)$ are present on the segment $[-\delta, \delta]$, and system (1.1) is stable outside the segment $[-\delta, \delta]$. Then we find that v_μ, and hence $p_\mu(t)$, are of the order of S, i.e. they do not exceed $\delta \max|c^*|$. The functions $F(t)$ and $\varphi(t)$ representing the linear combinations of $p_\mu(t)$ are also of the order of S, and since δ is small, will be much smaller than the characteristic parameter c^*. Thus under the assumptions made we can neglect in (1.2) the functions $\varphi(t)$ and $F(t)$ as compared with the remaining terms, i.e. assume that

$$F(t) = 0, \quad \varphi(t) = 0 \tag{2.1}$$

If there are no perturbations at the initial instant, apart from c^*, on the segment $[-\delta, \delta]$, then the area gain q will be determined by reflection of the perturbations v_μ of order S from the boundaries of the segment $x = -L_1$ and $x = L_2$ and from the inhomogeneities of the fundamental solution.

If the reflection coefficients are of the order of unity and the reflection occurs at a finite distance where the rate of propagation of the perturbations connected with the characteristics of the first family is of the order of unity, then the area flux of the reflected perturbation will be of the same order as its amplitude.

This means that q_i can be regarded as a linear functional of $S(t)$ such that the order of $q_i(t)$ does not exceed that of $\max_{0<\tau<t} S(\tau)\exp\alpha(t-\tau)$, remembering that the perturbations arriving at the boundaries of the segment $[-\delta, \delta]$ vary their areas near it as $\exp \alpha t$. We find here that $q \sim \alpha S$, and the sign of the right-hand side of (1.5) can be arbitrary, i.e. no general conclusions can be made concerning the behaviour of S.

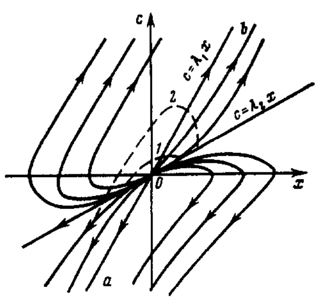

Fig.1

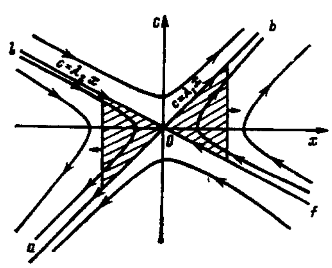

Fig.2

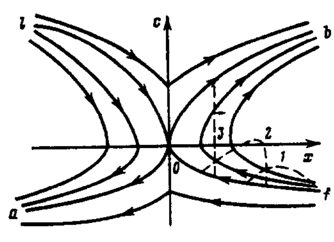

Fig.3

Nevertheless it will be shown that all conclusions concerning the instability, made in /1/, without taking q into account, remain valid when the term and the assumptions made here are both taken into account.

Note that the assumptions made are not unique. In some cases the area gain q may be much smaller than αS, e.g. when the reflection coefficients of the perturbations v_μ of the values of $C(x)$ at the points of reflection are small. In these cases the area gains need not be taken into account when investigating the stability, as was done in /1/.

3. As we have already noted, whenever the perturbation c^* leaves the segment $[-\delta, \delta]$, the boundary conditions at its ends need not be specified for c^* and the investigation of the behaviour of the perturbations near the critical points is the same as that carried out in /1/

In particular, the conclusion concerning the instability of the solution corresponding to any integral curve passing through a node-type singularity with positive characteristic directions in the x, c-plane (see Fig.1), is valid. The case arises when $\gamma = 0, \alpha > 0, \beta > 0$. The characteristic curves of the first family diverge from the critical point. The arrows in Fig.1 (as well as in Fig.2 and 3) show the directions in which the quantities c and x vary as t increases.

The perturbation of the stationary solution different from zero at $x = 0$, at the initial instant $t = 0$ (see Fig.1) remains always in the neighbourhood of the critical point and the derivative in x of the solution will tend, as $t \to \infty$, to the largest eigenvalue of the singularity i.e. to λ_1. The discontinuities, provided that they appear in the solution, will move away from the singularity either to the left, or to the right.

Let us consider the stationary solutions of (1.1) in the presence of a saddle-type singularity. In this case we have in (1.2) $\gamma = 0, \beta > 0$, and the quantity α can have any sign.

When $\alpha > 0$, the solution corresponding to the saddle separatrix, with characteristic curves converging to the critical point, remains unstable, within the assumptions made in Sect.2, even when q is taken into account. Indeed, let us select the perturbation c^* of the solution *lof* at $t = 0$ in such a manner, that the total area $S(0) = 0$, and the areas appearing on different sides of the cirtical point are different from zero (see Fig.2). Each area to the left and right of $x = 0$ will grow with time as $\exp \alpha t$, and the total area, and q together with it, will remain equal to zero. The presence of such increasing perturbations means that the stationary solution in question is unstable.

It is interesting to note that a solution, analogous to one obtained above under a series of assumptions for a non-linear system, also exists for a linear hyperbolic system of general type equations, provided that one family of characteristic curves converges to the critical point.

Let us now inspect the behaviour of the perturbations of the stationary solutions when $\gamma \neq 0$. The necessary condition for the stationary solution of (1.2), continuous and single-valued in x, to exist in the neighbourhood of the point $x = 0, c = 0$ is, that the quantity γ takes different values to the left and right of the critical point, and $\gamma_- < 0$ when $x < 0$, $\gamma_+ > 0$ when $x > 0$ /2/. The stationary solutions in the x, c-plane have the corresponding integral curves near the saddle compressed along the x-axis (Fig.3). The separatrices of this saddle are described near the origin of coordinates, to a first approximation, by the formulas $C(x) = \pm\sqrt{2\gamma_\pm x}$. It must be assumed, in general, that $\alpha_- \neq \alpha_+$; and since the coefficients γ and α are discontinuous, (1.2) must be considered separately to the left and right of the critical point, and this is indicated by the minus and plus sign respectively. When $\gamma \neq 0$, the term βx in (1.2) can be neglected compared with γ and αc.

The perturbations c^* of the solution *aob*, along which the characteristic curves diverge, leave the δ-neighbourhood of the critical point after a time of order $\sqrt{\delta}$.

The solution *lof* with converging characteristics is unstable if $\alpha_- = \alpha_+ > 0$. Here, just as in the dase of a saddle with a positive value of α, we can construct an increasing solution which does not generate the incoming perturbations. If the values of α are different on the left and right of the critical point, we cannot draw any general conclusions regarding the stability which would only be based on the development of the perturbations near the critical point. Nor can we draw any conclusions about the stability of the stationary solutions *lob* and *aof* for any $\alpha_\pm$, since there are no symmetrical solutions near the critical points whose total area is equal to zero.

We can, however, mention other cases in which the growth in the area of perturbations S near the critical point will lead to instability of the whole solution *lof*, *lob* or *aof* in toto.

Let us consider, to be specific, the behaviour of the perturbations of the solution *lof*. Let all characteristic velocities c^μ $(\mu = 2, 3, \ldots, n)$ have the same sign, e.g. positive, and $\alpha_- > 0$. Then the area of negative perturbations of the solutions *lof* between the integral curve *aol* and the discontinuity on the left will increase as $\exp \alpha_- t$, while the perturbations connected with the characteristics c^μ, will move away along the solution *lof* to the end of $x = L_1$, generating the perturbations c^*. However, only the negative perturbations c^* can pass through the critical point. The positive perturbations of magnitude c, which could compensate for the growth of the negative perturbations, will remain to the right of the point $x = 0$. Thus the area gain q can lead, at $x < 0$, to even faster growth of the perturbation of *lof*, i.e. to instability. Under these conditions the solution *lob* is also unstable.

If all characteristic velocities c^μ $(\mu = 2, 3, \ldots, n)$ are negative and $\alpha_+ > 0$, then the solutions *lof* and *aof* will also be unstable for the same reasons.

The instability of the solution *lof* will develop under the same conditions, and when the critical point of the saddle or deformed saddle-type, coincides with the right-hand end of the segment of the x-axis, i.e. for the solution *lo* when $c^\mu > 0, \alpha_- > 0$.

Thus the presence of the critical point in the stationary solution can lead to instability whether the characteristic curves diverge from or converge to the critical point, when the coefficient α in (1.2) is positive. Sect.3 lists the solutions for which the intrinsic specific instability is connected with the presence of the critical point.

The results can be extended to solutions of the parabolic degenerate systems of equations, provided that a characteristic velocity exists, which vanishes.

REFERENCES

1. KULIKOVSKII A.G. and SLOBODKINA F.A., On the stability of arbitrary steady flows in the neighbourhood of points of transition through the speed of sound. PMM Vol.31, No.4, 1967.
2. KULIKOVSKII A.G. and SLOBODKINA F.A., On the behaviour of small perturbations of one-dimensional steady transonic flows. PMM Vol.46, No.6, 1982.

Translated by L.K.